A Novel Parameter-dependent Input Normalization-based Direct MRAC with Unknown Control Direction

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***Abstract*— In this paper, we endow the model reference adaptive control (MRAC) with a novel parameter-dependent input normalization (PIN) to completely eliminate the conventional assumption of the high-frequency gain. Specifically, neither the sign nor the prior knowledge of the upper or lower bounds is required. To this end, we resort to an error augmentation together with a smart design of an adaptive law with a dead zone operation. Global stability in the mean square sense is established with the conventional proof concepts of the augmented error approach. In this way, no persistent excitation requirement is required. Although the system in question is assumed to be unity-relative-degree, the proposed technique can be easily extended to systems of arbitrary relative degrees. Finally, compared with the Nussbaum function-based methods in a numerical experiment, we show that transient behavior in our method is significantly improved.**

I. INTRODUCTION

We consider an uncertain LTI SISO system with a unitary relative degree described by

$$y(s) = k_p G(s) \llbracket u(t) \rrbracket, \tag{1}$$

with control input $u \in \mathbb{R}$, measurable output $y \in \mathbb{R}$ and high-frequency gain $k_p \in \mathbb{R}\setminus\{0\}$. The sign of k_p is referred to as the control direction. $G\llbracket u \rrbracket$ denotes the output of the operator $G(s)$ (in the Laplace domain) with input $u(t)$. The operator $G(s)$ is in the form of $G(s) = N(s)/D(s)$, where $N(s)$ and $D(s)$ are monic polynomials with degrees of $n-1$ and n respectively. The uncertainties lie in k_p, $N(s)$, $D(s)$, which are all completely unknown. The tracking problem under the MRAC framework pertains to the design of an output feedback controller for $u(t)$ such that the system output $y(t)$ tracks a reference signal y_r, which is generated from

$$y_r(s) = \frac{1}{s+p} \llbracket r(t) \rrbracket \tag{2}$$

with a piecewise-continuous and bounded reference input $r \in \mathbb{R}$ and $p > 0$.

The classical MRAC framework consists of a certainty equivalent controller whose parameter is updated online via an adaptive law, the latter based on output-error parameterization or input-error parameterization. Among various approaches [1]–[3] developed under MRAC framework, however, the prior knowledge of the sign of k_p is necessary and usually assumed to be known as prior. This assumption severely hampers the implementation of MRAC scheme to applications with unknown control direction, to name a few, uncalibrated robotics visual servoing [4], braking problem in ABS systems [5], and aircraft attitude tracking [6].

Consequently, removing the assumption of the sign of k_p in the MRAC framework is of great interest. There are three mainstream techniques in the literature to deal with the control direction uncertainty. The first one is using the Nussbaum approach [6]–[8], that consists of changing the adaptive gain with a smooth function whose sign is alternating in time, so as to explore different control directions. This technique is known to provide a poor transient behavior, which makes its practical implementation inadmissible [9], [10]. The second is using a switching signal to detect the control direction in real-time [11], [12]. Nevertheless, the discontinuous switching of the controller is prone to induce high-frequency chattering, which severely limits its application [4]. The third is in fashion of indirect adaptive control, and using the inverse of such an estimate in the formulation of the control input (such a technique requires the application of a switching projection [1] in the adaptation to avoid singularities upon inverting the estimated k_p). However, the erratic behavior brought by possible non-stop switching and the issue of unverifiable *persistence excitation* (PE) assumption on the regressor signal have not been solved yet. Remarkable is the recent work [10], in which under a weak interval excitation (IE) assumption, a dynamic regressor extension and mixing (DREM) estimator is employed and without any parameter projection and switching mechanism, asymptotic tracking is achieved. Unfortunately, even if the IE assumption is satisfied, there might occur parameter frozen phenomena [13]–[15] affecting the closed-loop stability. Therefore, a natural question arises: *Instead of the Nussbaum approach, how to remove the longstanding high-frequency gain assumption of MRAC, and not resort to the PE assumption on regressors or reference input?* This question, to the best of the authors' knowledge, remains open and seems far from settled.

In our previous work [16], we developed an unnormalized PIN-based MRAC scheme to directly estimate the unknown high-frequency gain without limitations of aforementioned schemes. The key point to overcome the possible singularity is injecting through unmeasurable derivatives of the output error with a switching gain function. However, to overcome

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the issue of avoiding the direct differentiation of the tracking error, we resorted to additional filtering stages that increased the complexity of the overall scheme and required invoking non-conventional proof concepts to prove global asymptotic stability of the adaptive system. Moreover, inherited from the flaws of unnormalized MRAC, the extension to systems with arbitrary relative degrees is not trivial, and the inaccessibility of higher derivatives of the output error restricts the algorithm in [16] to unity-relative-degree systems.

In this paper, we resort to an augmented error approach where unmeasurable derivatives of the output error do not represent an issue, being naively filtered by a causal operator. Notably, the classical adaptive law is modified with a dead zone function, which is core to preventing infrequent switching and ensuring the switching stops at finite times. Then, conventional proof concepts can be applied, and global stability in the mean square sense is proven. Also, the augmented error approach dramatically simplified the derivation of unavailable differentiation of the tracking error, and the dead zone in the adaptive law enhances the robustness of the overall adaptive system. Though the system considered in this paper is unity-relative-degree, the idea of the augmented error approach can be easily extended to systems with arbitrary relative degrees. To clarify its main idea, we only consider systems with a unitary relative degree here. The distinctive features of the proposed scheme are:

1) Neither the sign nor the prior knowledge of the upper or lower bounds is required. The conventional assumption of the high-frequency gain is completely eliminated.
2) Distinct from [16], the whole algorithm is explicitly simplified and renders the extension to systems with arbitrary relative degrees possible.
3) Without using any Nussbaum-type oscillating function, the transient behavior is substantially improved.
4) No persistence excitation or sufficiently rich requirement is required for the regressor and reference signals.

II. Problem Formulation

The MRAC problem is formally formulated in this section. In this work, we do not require prior knowledge of the sign or bound information of k_p, which is usually needed in the conventional MRAC framework [1].

Assumption II.1. The monic polynomial $N(s)$ is Hurwitz.

Assumption II.2. The degree n of monic polynomial $D(s)$ is known and the relative degree of the plant (1) is $\nu := 1$.

Define the tracking error as $\tilde{y}(t) := y(t) - y_r(t)$. Under the MRAC framework, we express the tracking error in Elliott's parameterization form [3] to avoid overparametrization and ignore the exponentially decaying effect of initial conditions

$$\tilde{y}(s) = k_p \frac{1}{s+p} \left[\!\left[u(t) - \xi^\top(t)\theta \right]\!\right] \tag{3}$$

where $\theta \in \mathbb{R}^{2n+1}$ is the unknown constant parameter vector and $\xi \in \mathbb{R}^{2n+1}$ is a vector of regressors generated by collating the reference signal with filtered input and output signals in the following form

$$\xi_u(s) = \frac{A(s)}{\Lambda(s)} [\![u(t)]\!] \in \mathbb{R}^n, \quad \xi_y(s) = \frac{A(s)}{\Lambda(s)} [\![y(t)]\!] \in \mathbb{R}^n,$$
$$\xi^\top(t) = \begin{bmatrix} \xi_u^\top(t) & \xi_y^\top(t) & r(t) \end{bmatrix} \in \mathbb{R}^{2n+1} \tag{4}$$

in which $A(s) := \begin{bmatrix} 1 & s & \cdots & s^{n-1} \end{bmatrix}^\top \in \mathbb{R}^n$, and $\Lambda(s)$ is an arbitrary monic and Hurwitz polynomial of order n.

For future use, we introduce the definition of μ-small in the mean square sense (m.s.s)[1] usually adopted in robust adaptive control to formulate our tracking problem.

Definition II.1. [1] Let $x : [0, \infty) \mapsto \mathbb{R}^n$, where $x \in \mathcal{L}_{2e}$, and consider the set

$$\mathcal{S}(\mu, t_0) = \left\{ x : [0,\infty) \mapsto \mathbb{R}^n \,\middle|\, \int_t^{t+T} x^\top(\tau) x(\tau) \mathrm{d}\tau \le c_0 \mu T + c_1, \forall t \ge t_0, T \ge 0 \right\}$$

for a given constant $\mu \ge 0$, where $c_0, c_1 \ge 0$ are some finite constants, and c_0 is independent of μ. We say that x is μ-small in the m.s.s for $t \ge t_0$ if $x \in \mathcal{S}(\mu, t_0)$.

In this study, we employ a deadzone operation aimed at mitigating potential instability stemming from continuous switching. While this approach enhances robustness, it comes at the cost of compromising the ideal property of asymptotic convergence and may introduce nonzero errors at steady state, as elucidated by the m.s.s. property of Definition II.1. Nonetheless, by selecting a suitably small dead-zone parameter, the magnitude of steady-state errors can be minimized.

Problem II.1. Under Assumptions II.1-II.2 and given *unknown* high-frequency gain k_p, polynomials $N(s), D(s)$ of the plant (1), for any reference signal $y_r(t) \in \mathcal{L}_\infty$, which is generated from the reference model (2) with reference input signal $r(t) \in \mathcal{L}_\infty$, the control objective is then to design an output feedback control input $u(t)$ such that

i) all the trajectories of the closed-loop system are bounded;
ii) for a given $\epsilon > 0$, there exists a $T_\epsilon > 0$, such that the tracking error $\tilde{y}(t)$ satisfies $\tilde{y}(t) \in \mathcal{S}(\epsilon, T_\epsilon)$.

III. The PIN-based MRAC scheme

With Assumptions II.1 and II.2, we are ready to introduce the PIN-based MRAC scheme to solve Problem II.1 without any prior information of the high-frequency gain k_p.

A. Controller design

Rewriting (3) into its state-space form $\dot{\tilde{y}} = -p\tilde{y} + k_p(u - \xi^\top\theta)$ and multiplying its both sides by $\beta := k_p^{-1}$ gives

$$\beta\dot{\tilde{y}} = -\beta p \tilde{y} + u - \xi^\top \theta. \tag{5}$$

Again, multiplying both sides by a gain function $\chi(t)$ to be determined later, it follows that

$$\chi\dot{\tilde{y}} = -\chi p \tilde{y} + \chi k_p u - \chi \xi^\top \vartheta \tag{6}$$

[1]In [1], $t \ge 0$ is used rather than $t \ge t_0$.

with $\vartheta := k_p\theta$. Adding (5) and (6) together leads to

$$(\beta+\chi)\dot{\tilde{y}} = -(\beta+\chi)p\tilde{y} + u + \chi k_p u - \xi^\top\theta - \chi\xi^\top\vartheta,$$

which implies the ideal control law (capable of regulating the tracking error to zero based on the full information of the plant parameters) takes the form of $u^* = (\xi^\top\theta + \chi\xi^\top\vartheta)/(1+\chi k_p)$. This simplified line of reasoning allows us to propose the implementable controller:

$$u = \frac{1}{1+\chi\hat{k}_p}\left(\xi^\top\hat{\theta} + \chi\xi^\top\hat{\vartheta}\right) \tag{7}$$

in which $\hat{k}_p \in \mathbb{R}$ and $\hat{\theta}, \hat{\vartheta} \in \mathbb{R}^{2n+1}$ are the estimates of k_p, θ and ϑ respectively. The estimated parameters and their convergence properties are given in the sequel.

B. Adaptive law and its convergence property

To derive the estimator of k_p, θ, ϑ, we first express (3) in time domain as $\tilde{y} = k_p(u_f - \xi_f^\top\theta)$ with $[\![u_f]\!] := \frac{1}{s+p}[\![u]\!]$ and $[\![\xi_f]\!] := \frac{1}{s+p}[\![\xi]\!]$. Employing the same manipulations as we did for (5)-(6), we multiply its both sides by β and χ respectively, and adding them together gives the expression

$$(\beta+\chi)\tilde{y} = u_f + \chi k_p u_f - \xi_f^\top\theta - \chi\xi_f^\top\vartheta. \tag{8}$$

Adding to both sides of (8) the term $\tilde{\beta}\tilde{y}$ with $\tilde{\beta} := \hat{\beta} - \beta$ and $\hat{\beta} \in \mathbb{R}$ as the estimate of β, then the error equation becomes

$$\tilde{y} = \frac{1}{\hat{\beta}+\chi}u_f + \frac{\tilde{y}}{\hat{\beta}+\chi}\tilde{\beta} + \frac{\chi u_f}{\hat{\beta}+\chi}k_p - \frac{\xi_f^\top}{\hat{\beta}+\chi}\theta - \frac{\chi\xi_f^\top}{\hat{\beta}+\chi}\vartheta. \tag{9}$$

Based on (9), we introduce an augmented error signal e_a as

$$e_a = \tilde{y} - \frac{1}{\hat{\beta}+\chi}u_f - \frac{\chi u_f}{\hat{\beta}+\chi}\hat{k}_p + \frac{\xi_f^\top}{\hat{\beta}+\chi}\hat{\theta} + \frac{\chi\xi_f^\top}{\hat{\beta}+\chi}\hat{\vartheta}, \tag{10}$$

from which a linear estimation error equation is obtained

$$e_a = \frac{\tilde{y}}{\hat{\beta}+\chi}\tilde{\beta} - \frac{\chi u_f}{\hat{\beta}+\chi}\tilde{k}_p + \frac{\xi_f^\top}{\hat{\beta}+\chi}\tilde{\theta} + \frac{\chi\xi_f^\top}{\hat{\beta}+\chi}\tilde{\vartheta} \tag{11}$$

with $\tilde{k}_p := \hat{k}_p - k_p$, $\tilde{\theta} := \hat{\theta} - \theta$ and $\tilde{\vartheta} := \hat{\vartheta} - \vartheta$.

According to (11), we resort to a normalized gradient-based estimation law with a dead zone as follows:

$$\begin{aligned} \dot{\hat{\beta}} &= -\frac{\gamma}{\hat{\beta}+\chi}\frac{\tilde{y}}{m}(\frac{e_a}{m}+g), & \dot{\hat{k}}_p &= \frac{\gamma\chi}{\hat{\beta}+\chi}\frac{u_f}{m}(\frac{e_a}{m}+g), \\ \dot{\hat{\theta}} &= -\frac{\gamma}{\hat{\beta}+\chi}\frac{\xi_f}{m}(\frac{e_a}{m}+g), & \dot{\hat{\vartheta}} &= -\frac{\gamma\chi}{\hat{\beta}+\chi}\frac{\xi_f}{m}(\frac{e_a}{m}+g), \end{aligned} \tag{12}$$

in which γ is an arbitrary positive constant, the augmented error signal e_a is defined in (10),

$$m := \left(1 + \left(\frac{\tilde{y}}{\hat{\beta}+\chi}\right)^2 + \left(\frac{\chi u_f}{\hat{\beta}+\chi}\right)^2 + \frac{(1+\chi^2)\xi_f^\top\xi_f}{(\hat{\beta}+\chi)^2}\right)^{\frac{1}{2}}, \tag{13}$$

and the dead zone function is designed as

$$g(t) = \begin{cases} 0, & \text{if } |\frac{e_a}{m}| \geq g_0 \\ -\frac{e_a}{m}, & \text{if } |\frac{e_a}{m}| < g_0 \end{cases} \tag{14}$$

with a user-defined threshold constant $g_0 > 0$.

To overcome the singularity problem possibly affecting the denominators in the control law (7) and the adaptive law (12), the gain function $\chi(t)$ is delicately designed as:

$$\chi = \hat{k}_p + \chi_h \tag{15}$$

with a hysteretic switching dynamics $\chi_h(t)$ designed to be

$$\chi_h(0) = \begin{cases} -1, & \hat{\beta}(0) + \hat{k}_p(0) < \frac{1}{2} \\ 1, & \hat{\beta}(0) + \hat{k}_p(0) \geq \frac{1}{2} \end{cases}$$

$$\chi_h(t) = \begin{cases} -1, & \hat{\beta}(t) + \hat{k}_p(t) \leq -\frac{1}{2} \\ 1, & \hat{\beta}(t) + \hat{k}_p(t) \geq \frac{1}{2} \\ x_h(t^-), & -\frac{1}{2} < \hat{\beta}(t) + \hat{k}_p(t) < \frac{1}{2} \end{cases}, t > 0,$$

where $\chi_h(t^-) := \lim_{\tau\to t^-}\chi_h(\tau)$ denotes the left-hand limit of the function χ_h at time t.

Lemma III.1. The gain function $\chi(t)$ in (15) guarantees that both $1 + \chi\hat{k}_p \neq 0$ and $\hat{\beta} + \chi \neq 0$ for all $\hat{k}_p$ and $\hat{\beta}$.

Proof. With χ in the form of (15), we obtain $1 + \chi\hat{k}_p = 1 + \hat{k}_p^2 + \hat{k}_p\chi_h$, $\hat{\beta} + \chi = \hat{\beta} + \hat{k}_p + \chi_h$. For the former, since $|\chi_h| \leq 1$, one derives $1 + \chi\hat{k}_p \geq \frac{3}{4}$. As for the latter, thanks to the chosen χ_h, one has $|\hat{\beta}+\chi| \geq \frac{1}{2}$. Hence, the singularity problem is overcome through the switching of χ. □

Remark III.1. Compared to the adaptive laws [1], [17], which overcome the singularity problem by switching the estimate $\hat{k}_p$ across the singular point, here the key idea of our method is to inject a signal $\chi\dot{\tilde{y}}$ through the input u into (5). Then, thanks to the parameter-dependent gain function χ, we are able to normalize the input u, which solves the singularity issue. Besides, with the augmented estimates $\hat{\beta}, \hat{k}_p, \hat{\theta}, \hat{\vartheta}$, we derive a linear error estimation equation in (11), thus bypassing the division of $\hat{k}_p$ and meanwhile relaxing the prior knowledge of the lower boundary of k_p.

Lemma III.2. Consider the linear estimation error equation (11) and the gain function χ in (15), the adaptive law (12) with the dead zone (14) guarantees that $\hat{\beta}, \hat{k}_p, \hat{\theta}, \hat{\vartheta} \in \mathcal{L}_\infty$ and $\frac{e_a}{m} \in \mathcal{S}(g_0^2, 0) \cap \mathcal{L}_\infty$ and $\dot{\hat{\beta}}, \dot{\hat{k}}_p, \dot{\hat{\theta}}, \dot{\hat{\vartheta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

Proof. Consider the Lyapunov-like function $V = (\tilde{\beta}^2 + \tilde{k}_p^2 + \tilde{\theta}^\top\tilde{\theta} + \tilde{\vartheta}^\top\tilde{\vartheta})/(2\gamma)$, whose derivative along the trajectory of (12) can be immediately computed as

$$\begin{aligned} \dot{V} &= -\frac{\tilde{y}\tilde{\beta} - \chi u_f\tilde{k}_p + \xi_f^\top\tilde{\theta} + \chi\xi_f^\top\tilde{\vartheta}}{m(\hat{\beta}+\chi)}\left(\frac{e_a}{m}+g\right) \\ &= -\frac{e_a}{m}(\frac{e_a}{m}+g) \\ &= \begin{cases} 0, & \text{if } |\frac{e_a}{m}| < g_0 \\ -(\frac{e_a}{m})^2, & \text{if } |\frac{e_a}{m}| \geq g_0 \end{cases} \end{aligned} \tag{16}$$

implying $\dot{V} \leq 0$. Hence, $\hat{\beta}, \hat{k}_p, \hat{\theta}, \hat{\vartheta} \in \mathcal{L}_\infty$. Then, from (11) and (13), we obtain $\frac{e_a}{m} \in \mathcal{L}_\infty$ due to the boundedness of

$\hat{\beta}, \hat{k}_p, \hat{\theta}, \hat{\vartheta}$ and the normalization by m. It, in turn, implies $\dot{\hat{\beta}}, \dot{\hat{k}}_p, \dot{\hat{\theta}}, \dot{\hat{\vartheta}} \in \mathcal{L}_\infty$ from (12). From (16), it follows

$$\dot{V} = -\frac{e_a}{m}(\frac{e_a}{m} + g) \leq -(\frac{e_a}{m})^2 + |\frac{e_a}{m}|g_0 \leq -\frac{1}{2}(\frac{e_a}{m})^2 + \frac{g_0^2}{2}$$

which implies that $\frac{e_a}{m} \in \mathcal{S}(g_0^2, 0)$.

Let's examine the $\mathcal{L}_2$ property of the parameter estimation derivatives. From (14), it is trivial to obtain $g(\frac{e_a}{m} + g) = 0$, thus $(\frac{e_a}{m} + g)^2 = \frac{e_a}{m}(\frac{e_a}{m} + g) + g(\frac{e_a}{m} + g) = \frac{e_a}{m}(\frac{e_a}{m} + g)$. Then, utilizing the integrable property of $\dot{V}$, we derive $\frac{e_a}{m} + g \in \mathcal{L}_2$, which, from (12), implies $\dot{\hat{\beta}}, \dot{\hat{k}}_p, \dot{\hat{\theta}}, \dot{\hat{\vartheta}} \in \mathcal{L}_2$. □

Due to Lemma III.2, the boundedness of $\dot{\hat{\beta}}$ and $\dot{\hat{k}}_p$ ensures the existence of an upper bound of $|\dot{\hat{\beta}} + \dot{\hat{k}}_p|$, which implies the existence of a non-zero dwell time $\tau > 0$ such that

$$t_{k+1} - t_k \geq \tau, \quad \forall k \in \{0, 1, 2, \cdots\} \tag{17}$$

in which the switching instant t_k determines the change the sign of χ_h. It is explicitly seen that the dead zone destroys the ideal convergence property of the adaptive law, such as $\frac{e_a}{m} \in \mathcal{L}_2$, however it plays a critical role in preventing infrequent switching of (15), that may result in instability (as claimed in Remark 3.2 in [18]). Potentially, if there is a measurement noise in the output y, the dead zone will prevent parameter drifting and improve the robustness of the proposed algorithm. Moreover, thanks to the introduction of the dead zone, the switching will stop after some finite time, which is concluded in the following lemma:

Lemma III.3. Consider the gain function χ in (15), there exists a finite time $T_f > 0$ such that the switching stops with a finite switching number for all $t \geq T_f$.

Proof. From (16), we have $\dot{V} = -\frac{e_a}{m}(\frac{e_a}{m} + g) = -|\frac{e_a}{m}||\frac{e_a}{m} + g|$, where the second equation holds due to the choice of g in (14). Since $|\frac{e_a}{m} + g| = 0$ if $|\frac{e_a}{m}| < g_0$, it follows that $\dot{V} = -|\frac{e_a}{m}||\frac{e_a}{m} + g| \leq -g_0|\frac{e_a}{m} + g|$, which implies that $\frac{e_a}{m} + g \in \mathcal{L}_1$. In view of (12), we have $|\dot{\hat{\beta}}| \leq \gamma|\frac{e_a}{m} + g|$, $|\dot{\hat{k}}_p| \leq \gamma|\frac{e_a}{m} + g|$, where we utilize the normalization of m. Thus, $\dot{\hat{\beta}}, \dot{\hat{k}}_p \in \mathcal{L}_1$, which in turn implies the existence of

$$\lim_{t\to\infty} \int_0^t \dot{\hat{\beta}} \mathrm{d}\tau = \lim_{t\to\infty} \hat{\beta}(t) - \hat{\beta}(0)$$
$$\lim_{t\to\infty} \int_0^t \dot{\hat{k}}_p \mathrm{d}\tau = \lim_{t\to\infty} \hat{k}_p(t) - \hat{k}_p(0),$$

and $\hat{\beta}(t), \hat{k}_p(t)$ converge to some limits, denoted as $\hat{\beta}_0, \hat{k}_{p0}$ respectively (see [1, Chapter 8.4.3] for more technical details). Hence, $\hat{\beta}(t) + \hat{k}_p(t)$ converge to its limit $\hat{\beta}_0 + \hat{k}_{p0}$. Then, with the choice of χ_h designed in (15), the switching must stop after some finite time T_f, and moreover, the switching number is finite due to the finite dwell time and T_f. □

Remark III.2. The control direction uncertainty, in the existing literature [1], [8], [17], could be overcome by utilizing Nussbaum-type functions or adopting a switching projection. Unfortunately, the former has an inferior transient behavior which is practically inadmissible while the latter possesses an erratic behavior brought by possible non-stop switching. It is worth noting that the DREM-based scheme [17] guarantees at most one switching, however, among these switching projection-based methods, the issue of unverifiability PE assumption on the regressor signal has not been solved yet. In our method, we guarantee there exists a finite switching and relax the PE condition for the first time.

IV. Stability Analysis

Now, we proceed with the stability analysis of the closed-loop system under the control protocol (7).

Theorem IV.1. With Assumptions II.1-II.2, the controller (7) composed of the adaptive law (12) and the gain function (15), there exists a positive constant g_0^* such that for all $g_0 < g_0^*$, the closed-loop trajectories are bounded and the tracking error $\tilde{y}$ satisfies $\tilde{y} \in \mathcal{S}(g_0^2, T_f)$ for some finite time T_f.

Proof. The main idea of the proof relies on the following observation: the linear estimation error equation (11) is obtained by the usual augmented error approach from (9)-(11), thus one can intuitively use standard results from literature [1], [2] that relate the augmentation term with the derivative of parameters. The non-standard thing, observe (10), is that some of parameters $\hat{k}_p, \hat{\vartheta}$ are multiplied by χ, then we must consider the weak derivative of these terms since χ is not time-differentiable at some points. At this point, the finite switching in Lemma III.3 plays a key role, that is, the finite number of switching implies that $\dot{\chi} \in \mathcal{L}_2$. Then, the proof of the usual augmented error approach still holds.

For future use, reverting back to (10), we define $\hat{\theta}_a := \begin{bmatrix} -1 & -\chi\hat{k}_p & \hat{\theta}^\top & \chi\hat{\vartheta} \end{bmatrix}^\top \in \mathbb{R}^{4n+4}$, $\omega_f := \begin{bmatrix} u_f & u_f & \xi_f^\top & \xi_f^\top \end{bmatrix}^\top \in \mathbb{R}^{4n+4}$, $\omega := \begin{bmatrix} u & u & \xi^\top & \xi^\top \end{bmatrix}^\top \in \mathbb{R}^{4n+4}$, then (10) becomes

$$e_a = \tilde{y} + \frac{1}{\hat{\beta} + \chi} \hat{\theta}_a^\top \omega_f. \tag{18}$$

Note that, by the definition of u in (7), it follows

$$\hat{\theta}_a^\top \omega = -(1 + \chi\hat{k}_p)u + (\xi^\top\hat{\theta} + \chi\xi^\top\hat{\vartheta}) = 0, \tag{19}$$

which is the key point of the following proof.

The proof contains three major steps:

First, to relate $\hat{\theta}_a^\top\omega$ with $\hat{\theta}_a^\top\omega_f$, we invoke the swapping lemma [1], which with use of (19) establishes that

$$\frac{1}{s+p}\left[\!\left[\hat{\theta}_a^\top\omega\right]\!\right] = \left[\!\left[\hat{\theta}_a^\top\omega_f\right]\!\right] - \frac{1}{s+p}\left[\!\left[\dot{\hat{\theta}}_a^\top \frac{1}{s+p}[\![\omega]\!]\right]\!\right] = 0$$

which implies the following equation with (18):

$$[\![\tilde{y}]\!] = [\![e_a]\!] - \frac{1}{\hat{\beta}+\chi}\frac{1}{s+p}\left[\!\left[\dot{\hat{\theta}}_a^\top \frac{1}{s+p}[\![\omega]\!]\right]\!\right], \tag{20}$$

in which e_a can be expressed as $e_a = \frac{e_a}{m}m$ with $\frac{e_a}{m} \in \mathcal{S}(g_0^2, 0)$ from Lemma III.2.

Second, we introduce a fictitious signal[2] m_f as $m_f^2 := 1 + \|\tilde{y}\|_{2\delta}^2$, and in view of (20), m_f^2 satisfies

$$m_f^2 \leq 1 + \|\frac{e_a}{m}m\|_{2\delta}^2 + c\|\dot{\hat{\theta}}_a^\top\omega_f\|_{2\delta}^2 \tag{21}$$

[2] $\mathcal{L}_{2\delta}$ norm is defined as $\|x_{[0,t]}\|_{2\delta} := \left(\int_0^t e^{-\delta(t-\tau)}x^\top(\tau)x(\tau)d\tau\right)^{\frac{1}{2}}$.

where $c \geq 0$ is used to denote any finite constant. With (21), we will establish the relation between m, ω_f with $\tilde{y}$. Observing (13), the constitutions of m are $u_f, \xi_f, \tilde{y}$ which can be expressed as

$$\llbracket u_f \rrbracket = \frac{1}{s+p} \left\llbracket \frac{\hat{\theta}^\top + \chi \hat{\vartheta}^\top}{1 + \chi \hat{k}_p} \xi \right\rrbracket, \quad \llbracket \xi_f \rrbracket = \frac{1}{s+p} \llbracket \xi \rrbracket,$$
$$\llbracket \tilde{y} \rrbracket = \frac{k_p}{s+p} \left\llbracket \left(\frac{\hat{\theta} + \chi \hat{\vartheta}}{1 + \chi \hat{k}_p} - \theta \right)^\top \xi \right\rrbracket,$$
$$\llbracket \xi \rrbracket = \left[\frac{A(s)^\top G(s)^{-1}}{k_p \Lambda(s)} \llbracket \tilde{y} + y_r \rrbracket \quad \frac{A(s)^\top}{\Lambda(s)} \llbracket \tilde{y} + y_r \rrbracket \quad \llbracket r \rrbracket \right]^\top.$$

Since the operators $\frac{G(s)^{-1}}{s+p}$, $\frac{A(s)^\top G(s)^{-1}}{\Lambda(s)}$ and $\frac{A(s)^\top}{\Lambda(s)}$ are proper and stable and $\frac{1}{s+p}$ is strictly proper and stable, we utilize [1, Lemma 3.3.2] to derive $\frac{m}{m_f} \in \mathcal{L}_\infty$. Similarly, m_f guarantees that $\frac{\omega_f}{m_f} \in \mathcal{L}_\infty$. Therefore, (21) becomes $m_f^2 \leq 1 + c\|\tilde{\Theta} m_f\|_{2\delta}$ and can be rewritten into

$$m_f^2 \leq 1 + c \int_0^t e^{-\delta(t-\tau)} \tilde{\Theta}^2(\tau) m_f^2(\tau) \, \mathrm{d}\tau \tag{22}$$

where $\tilde{\Theta}^2 := (\frac{e_a}{m})^2 + \|\dot{\hat{\theta}}_a\|^2$.

The **last** thing is to utilize the *small-gain lemma* to show the boundedness of closed-loop trajectories and the convergence property of $\tilde{y}$. Hence, from (22), the key point is to show $\dot{\hat{\theta}}_a \in \mathcal{L}_2$, that is $\dot{\chi} \in \mathcal{L}_2$. Employing *triangle inequality*, the $\mathcal{L}_2$ norm of $\dot{\chi}$ is

$$\begin{aligned} \|\dot{\chi}\|_{\mathcal{L}_2}^2 &\leq \|\dot{\hat{k}}_p\|_{\mathcal{L}_2}^2 + \|\dot{\chi}_h\|_{\mathcal{L}_2}^2 \\ &\leq \|\dot{\hat{k}}_p\|_{\mathcal{L}_2}^2 + \int_0^\infty (\delta_h(t_k)\delta(t - t_k))^2 \, \mathrm{d}t \leq \|\dot{\hat{k}}_p\|_{\mathcal{L}_2}^2 + 4\bar{k} \end{aligned}$$

where t_k is the switching instant, and $\bar{k}$ is denoted as the maximum switching number, which is finite and proven in Lemma III.3, while $\delta(\cdot)$ is the Dirac's delta operator, $\delta_h(t) := \mathrm{sign}(\chi_h(t) - \lim_{\tau \to t^-} \chi_h(\tau))$ is a left-discontinuous function that encodes the up-fronts or down-fronts of χ_h. Thanks to $\dot{\hat{k}}_p \in \mathcal{L}_2$ from Lemma III.2 and the finite switching from Lemma III.3, one obtains $\|\dot{\chi}\|_{\mathcal{L}_2} < \infty$, that is $\dot{\hat{\theta}}_a \in \mathcal{L}_2$. Applying the *Bellman-Gronwall lemma III* [1, Lemma 3.3.9] to (22), one obtains $m_f^2(t) \leq \Phi(t, 0) + \delta \int_0^t \Phi(t, \tau) \, \mathrm{d}\tau$, in which $\Phi(t, \tau) = e^{-\delta(t-\tau)} e^{c \int_\tau^t \tilde{\Theta}^2(s) \, \mathrm{d}s}$. Since $\dot{\hat{\theta}}_a \in \mathcal{L}_2$ and $\frac{e_a}{m} \in \mathcal{S}(g_0^2, 0)$ by Lemma III.2, $\Phi(t, \tau)$ can be written into

$$\Phi(t, \tau) \leq c \, e^{-(\delta - c g_0^2)(t - \tau)}. \tag{23}$$

Hence, from (23), if g_0 is chosen such that $g_0^2 < \frac{\delta}{c} < g_0^{*2} := \frac{2p}{c}$, then $\Phi(t, \tau)$ is bounded from above by a decaying to zero exponential, which in turn, implies $m_f \in \mathcal{L}_\infty$. Hence, we have $\tilde{y}, m, \omega_f \in \mathcal{L}_\infty$ and all signals are bounded. After $t \geq T_f$, from Lemma III.3, the switching stops, that is $\dot{\chi}_h(t) = 0$. Then, resorting to (20) and with $m, \omega_f \in \mathcal{L}_\infty$, we obtain

$$\int_t^{t+T} \tilde{y}^2 \, \mathrm{d}\tau \leq c \int_t^{t+T} (\frac{e_a}{m})^2 \, \mathrm{d}\tau + c \int_t^{t+T} \|\dot{\hat{\theta}}_a\|^2 \, \mathrm{d}\tau \tag{24}$$

for all $t \geq T_f$ and $T \geq 0$. Again, with $\frac{e_a}{m} \in \mathcal{S}(g_0^2, 0)$ from Lemma III.2, we derive from (24) the inequality $\int_t^{t+T} \tilde{y}^2 \, \mathrm{d}\tau \leq c g_0^2 T + c$. Hence, $\tilde{y} \in \mathcal{S}(g_0^2, T_f)$. $\square$

V. Simulation

This section illustrates the effectiveness of the proposed PIN-based MRAC scheme compared with two algorithms, the first using a classical Nussbaum-gain [1, Chp 6.5.3], and the second employing a DREM with a switching projection [17]. Note that, since we utilize an augmented error approach, it is trivial to extend our algorithm to systems with a higher relative degree. Here we only exhibit the performance of the proposed method without resorting to mathematical developments that would unnecessarily overcomplicate the exposition and blur up the paper contribution. All the simulations are conducted under $ode45$ with the identical simulation precision of 1×10^{-3}.

We consider a system with a unitary relative degree in the form of

$$y = \begin{cases} -\dfrac{s+1}{s^2 + 2.1s + 0.2} \llbracket u \rrbracket, & t < 400 \\ \dfrac{s+1}{s^2 + 2.1s + 0.2} \llbracket u \rrbracket, & t \geq 400 \end{cases}$$

in which a sudden change of the sign of the high-frequency gain happens at 400 seconds, definitely posing a tough challenge. The measurement output $y(t)$ is to track the reference signal $y_r(t)$ generated by $y_r = \frac{1}{(s+1)} \llbracket r \rrbracket$ with the reference input $r(t) = \sin(3t)$, that is not sufficiently rich.

The filter $\Lambda(s)$ is chosen as $\Lambda(s) = (s+1)^2$. A sufficiently small threshold of the dead zone is selected as $g_0 = 0.001$. For the sake of a fair comparison, the adaptive gain in all three methods is chosen the same, i.e., $\gamma = 15$. In the DREM-based method, we select the tuning gain $\underline{k_p} = 0.1$ and operators $H_i(s) = \frac{i}{s+i}, i = 1, \cdots, 5$ while in the Nussbaum-based method, the Nussbaum-type function is chosen as $N(x) = x^2 \cos(x)$. The initial conditions of the reference model and filters are set to zero and the plant is initialized to be $y(0) = 2$. All the estimators are initialized to be zero except the $\hat{k}_p(0)$ set is chosen to be $\hat{k}_p(0) = 1$ in the DREM-based method, which is non-zero.

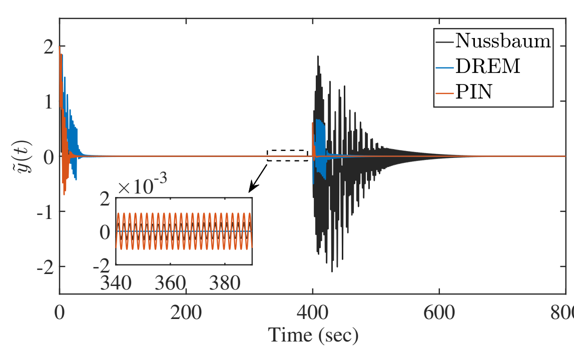

Fig. 1. Comparison of the time history of tracking error $\tilde{y}(t)$.

Fig. 1 demonstrates that all methods achieve fast-tracking of the output, but their transient performances behave differently, especially during the sudden change of high-frequency gain. Only one switching, as depicted in Fig. 2, happens such that the input by the proposed method can quickly respond to the sudden change and the smallest overshoot

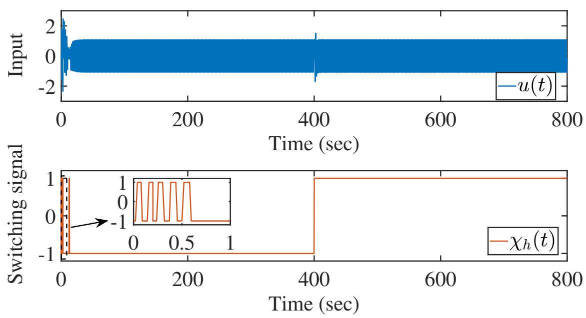

Fig. 2. Time history of the control input $u(t)$ and switching signal $\chi_h(t)$ by the PIN-based MRAC scheme.

occurs in the tracking error, which explicitly outperforms other methods. Due to the dead zone, the tracking error by the proposed method can only converge to a small residual set in the steady state, as shown in Fig. 1. However, the finite switching enables the proposed controller to overcome the control direction uncertainty at a very fast speed.

Next, we show the robustness of the proposed method with the measurable output as $y_a = y + d$, where d is a biased sinusoidal signal as $d(t) = 0.1 + 0.001\sin(100t)$. Though the amplitude of d is quite small, the unmodeled noise still incurs the performance deterioration of the Nussbaum-based method, while thanks to the dead zone, the tracking error by the proposed method pertains to satisfactory performance.

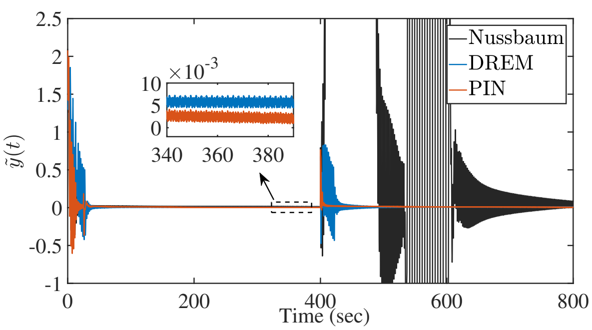

Fig. 3. Comparison of the time history of tracking error $\tilde{y}(t)$ under measurement noises.

VI. Conclusions

This paper has developed a different formulation of the PIN technique to deal with the unknown control direction under the MRAC framework. Distinguishing from [16], we create a new augmented error, and filter the unmeasurable output error. As a consequence, the whole algorithm is explicitly simplified. On the other hand, by invoking an augmented error approach, we prove the closed-loop stability with the small-gain lemma. Thus, no persistent excitation requirement is needed. Distinguishing from existing works of MRAC under unknown control direction, we completely remove the bottleneck assumption on the prior knowledge of the high-frequency gain, including the sign and the lower or upper bounds. The transient behavior is significantly improved by avoiding any Nussbaum gain-like oscillation function, which is shown in the numerical experiment. Also, the simulation results show enhanced robustness of the algorithm, which is more suitable for practical applications. Though the system considered is unity-relative-degree, by following the proposed design procedures, the extension to systems with arbitrary relative degrees and the multivariable version is trivial, which will be our future work.

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